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ATTEMPTS MADE DURING THE EIGHTEENTH AND NINETEENTH CENTURIES TO REFORM THE TEACHING OF GEOMETRY.*

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BIBLIOGRAPHY.

We have found the following six books bearing on the history of the teaching of geometry most useful in making this compilation:

1. V. BOBYNIN—"Elementare Geometrie," being Chapter XXII. in Cantor's *Vorlesungen über Geschichte der Mathematik,* Vol. IV, Leipzig, 1908, pp. 321-402. (Covers second half of eighteenth century.) Referred to as "Bobynin."

2. F. KLEIN—*Elementarmathematik vom Hoeheren Standpunkte aus, Theil II; Geometrie.* Leipzig, 1909, pp. 433-515. Referred to as "Klein."

3. J. PERRY—*Discussion on the Teaching of Mathematics.* British Association Meeting at Glasgow, 1901, London. Referred to as "Perry."

4. H. SCHOTTEN—*Inhalt und Methode des Planimetrischen Unterrichts.* Leipzig, 1890. Referred to as "Schotten."

5. M. SIMON—*Ueber die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert.* Leipzig, 1906. Referred to as "Simon."

6. A. W. STAMPER—*A History of the Teaching of Elementary Geometry,* New York, 1906. Referred to as "Stamper."

Other useful sources of information on the history of the teaching of geometry are as follows:

1. *Reports of the Association for the Improvement of Geometrical Teaching* (in England). The Association now calls itself "The Mathematical Association" and its present organ is the *Mathematical Gazette.*

2. *Zeitschrift für Mathematischen and Naturwissenschaftlichen Unter-*

*This article is a part of the report of the National Committee of Fifteen on a Geometry Syllabus. The Committee has been at work for nearly two years under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences. The Committee is not yet ready to present its report but feels that this historical setting prepared by Professor Cajori should be in the hands of mathematical teachers at once. THE EDITORS.

richt, Leipzig und Berlin. Formerly called "Hoffmann's Zeitschrift," now "Schotten's Zeitschrift."

3. *L'Enseignement Mathématique. Revue internationale.* Paris.
4. *Nature* (London). See Indexes for "Geometry."
5. G. LORIA — *Vergangene und Künftige Lehrplane.* Deutsch von H. WIELEITNER, Leipzig, 1906.
6. G. LORIA — *Della varia fortuna di Euclide,* Roma, 1893,
7. DODGSON — *Euclid and His Modern Rivals,* London, 1885 (2. ed.)
8. R. FRICKE. "Ueber Reorganisationsbestrebungen des Mathematischen Elementarunterrichts in England." *Jahresber. d. deutsch. Math. Vereinigung,* Vol. 13, 1904, p. 283, etc.
9. KLEIN-SCHIMMACK, *Vortrage über den mathematischen Unterricht an den Hoeheren Schulen,* Leipzig, 1907.
10. *Plan d'études et programmes d'enseignement dans les lycées et collèges de garçons.* Paris, 1903.
11. A. W. Stamper's list of references at the end of his book.
12. Histories of mathematics.

FRANCE.

France began to maintain a critical attitude toward Euclid as a geometrical text-book for beginners as early as the time of Petrus Ramus (1580). Ramus treated geometry as the art of accurate measurement. In the eighteenth century this spirit of independence was intensified by the publication of Clairaut's *Elémens de Géometrie* (1741), in which surveying and other practical matters received marked attention. In the latter half of the eighteenth century Euclid ceased to be used as a text-book in France.

Williamson, in his edition of Euclid, 1781, criticises Clairaut as follows: "Elements of geometry carefully weeded of every proposition tending to demonstrate another; all lying so handy that you may pick and choose without ceremony. 'This is useful in fortification;' 'you cannot play at billiards without this.' 'You only look through a telescope like a Hottentot until this proposition is read,' with many such powerful strokes of rhetoric to the same purpose. And upon such terms, and with such inducements, who would not be a mathematician? Who would go to work with all that apparatus which I have described as necessary for understanding Euclid, when he has only to take a pleasant walk with Clairaut upon the flowery banks of some delightful river, and there see, with his own eyes, that he must learn to draw a perpendicular before he can tell how broad it is?" About 1836 De Morgan remarks that these arraignments are not "without their force, when directed against experimental geometry as an ultimate course of study, [but] lose their ironical character and become serious earnest, when applied to the same as a preparatory method." De Morgan strongly favors a geometry like Clairaut's as a preparatory course.

The critical attitude of Ramus and Clairaut toward the *Elements* of

Euclid brought to the mind of D'Alembert the questions: What are the elements of a science? What should be the contents of a book called elements? D'Alembert gives his answers in two articles, "Elémens des sciences," and "Des élémens de géometrie" in the *Encyclopédie méthodique* (about 1784).*

D'Alembert distinguishes between two kinds of elements of a science:

(1) If all truths or theorems of a science which are the foundation for all others, are brought together, so that these truths or theorems potentially comprise the whole science, then these constitute, when properly coordinated, the *elements* of the science. In geometry, such elements embrace not merely the principles of mensuration and the properties of plane figures, but also the application of algebra to geometry, and the differential and integral calculus in its application to curved lines.

(2) The elements of a science may be defined also as comprising those truths or theorems which treat the subject matter in the simplest way, and which constitute, together with their deductions, a detailed study of the simplest parts of the science. By the elements of geometry, elements of this kind are usually meant; they include only the properties of plane figures and the circle.

Dissatisfied with the elements of geometry known in his day, D'Alembert sets up the following demands which such texts should fulfill:

(1) The text should develop the subject along the path pursued by the discoverers of the science; so as to show the truths in their natural relations to each other.

(2) The usual division of the subject into longimetry, planimetry and stereometry does not provide for the circle and sphere, and is therefore inadequate. The division into plane geometry and solid geometry, D'Alembert does not consider at all. He suggests the division into the geometry of the straight lines (considered with respect to position and relative magnitude) and circles, the geometry of surfaces and the geometry of solids. The straight line and circle must be taken up together. The circle renders immense service in considering the position of lines. The measurement of angles by circular arcs and the principle of congruence constitute the basis of the first part of the geometry of lines, upon which other theorems of this part rest. The second part in the geometry of the straight line has as its fundamental theorem the one on the section into proportional parts of two sides of a triangle by a line parallel to the third side. This involves incommensurables.

(3) Incommensurable relations must be treated by the apagogic method, according to which it is shown that one ratio cannot be greater or smaller than a certain other ratio, hence it must be equal to that other ratio. He uses this for the following reasons: Incommensurable magnitudes in-

***Encyclopédie methodique, Mathématiques I,* 617-625; III, 133-136. We have used the Italian translation of this dictionary, Padova, 1800, and also a full abstract of these articles, given by Bobynin. See Bobynin, p. 325, etc.

volve the idea of the infinite and therefore, he claims, cannot be treated by any direct method. Notwithstanding this difficulty presented by incommensurable lines, he maintains that they should be taken up early in geometry, because of their importance. He states that the whole theory of incommensurables demands only one theorem, concerning the limits of quantities, viz: "Magnitudes which are the limits of one and the same magnitude, or magnitudes which have one and the same limit, are equal to each other." In the geometry of the circle, of surfaces and solids, he feels that the method of exhaustion or that of limits should be used.

(4) A suitable text on the elements of geometry can be prepared only by a mathematician of the first rank. D'Alembert complains that most elementary geometries are written by men of little ability.

(5) To lay down definitions at the beginning without any analysis of the subject is not only contrary to sound philosophy but contrary to the natural march of thought.* Axioms are useless.

Ideas similar to those of D'Alembert are embodied in a text on geometry by Louis Bertrand of Genève,† who in Berlin had been close to Euler. Bertrand's book antedated D'Alembert's articles in the Encyclopédie Méthodique. Like D'Alembert he divides geometry into three parts: (1) Geometry of line and circle, (2) Measurement of parts of a plane bounded by straight lines and circles, (3) Measurement of curved surfaces and solids. Bertrand ignored the classification of geometry into plane and solid. His second theorem is: "When two planes intersect, their common section is a right line." The straight line and circle are taken up together at the beginning as D'Alembert would have it. The incommensurable case is treated by the *reductio ad absurdum* method. In the latter part of the geometry he uses also the method of exhaustion. Bertrand reduces the number of theorems, in one instance, by replacing theorems on the mensuration of prisms, pyramids, cylinders, cones, and spheres by the corresponding problems.

Bertrand's work was published in two unwieldy volumes and had little sale, yet exercised some influence, particularly upon Lacroix, whose *Cours de Mathématiques*, published at the close of the eighteenth century, has been used until recently. Lacroix divides his geometry into geometry of the plane and geometry of space, and does not follow D'Alembert closely. According to Lacroix there are only two kinds of theorems that should find a place in an elementary geometry: (1) Theorems necessary for the comprehension of the line of argument, developed synthetically. (2) Theorems which grow out of the practical operations in geometry (drawing and measuring). He objects to placing all axioms at the beginning, believes in the omission of the definition of an angle, favors "a straight line is the

*See "Axiome" and "Courbe" in *Encycl. Méth.*

†*Dévelopment nouveau de la partie élémentaire des mathématiques*, Genève, 1778.

shortest path between two points'' as growing out of the child's experience, and uses the apagogic method for incommensurables.

Another author of note was Bézout, who followed D'Alembert's plan quite closely, but was criticised for his lack of rigor and for his endeavor to lighten the work of the examiner as well as of those being examined.*

The most celebrated work on elementary geometry is that of Legendre (1794). He came nearest to fulfilling D'Alembert's requirement that the elements be written by a mathematician of the first rank. He does not follow D'Alembert's plan for a book on geometry, nor does he heed the philosophic demand that the author should follow the path of the originators of the science. Impressed by the lack of rigor in the works of his day, he aims at greater rigor and approaches closer to Euclid than his predecessors had been. He does not divide geometry in the manner of D'Alembert and Bertrand. Like Euclid, Legendre begins with definitions and axioms. The first four chapters are given to plane geometry, the last four to solid. The first book treats of the equality of angles and triangles, the second of the circle and the measurement of angles, the third of proportional figures, the fourth of regular polygons and the measurement of the circle. Legendre uses in measurement the terms *equal* and *equivalent.* He uses the *reductio ad absurdum* method for incommensurables and the method of exhaustion for curved lines.

What was it that made this book so successful? In the first place must be mentioned his great clearness of exposition and his attractive style. A great advance of Legendre over Euclid was the fuller treatment of solid geometry. He leans less toward logic and more toward intuition than does Euclid. In place of Euclid's famous fifth book on incommensurables, Legendre borrows rational and irrational numbers from arithmetic, even though in arithmetics no scientific treatment of those subjects was given in his day. A theorem true for rationals is assumed to be true for irrationals. Thus, if $A:B=C:D$, then $AD=BC$ in all cases. Klein says that this is in accordance with the practice of the best mathematicians of his day, that even Lagrange works out the expansion of $(x+h)^n$, when n is rational and assumes the results thus obtained to be true for irrational values of n. Legendre stands for a fusion of geometry, not only with arithmetic, but also with trigonometry. As late as 1845 Legendre's geometry still contained trigonometry, but as Klein remarks,† the trigonometry and the practical application of geometry were gradually filtered out. Comparing A. Blanchet's edition of 1876 with an edition of 1817, we find also that the twelve ''notes'' on topics of elementary geometry, covering 55 pages in the older edition, are omitted in the later edition. The later edition has a somewhat fuller treatment of solid geometry and a list of exercises in original proofs, loci and constructions. Other notable changes were made in the

*Bobynin, p. 355.

†Klein, p. 470.

1845 edition by J. B. Balleroy and A. L. Marchand. They state that Legendre uses the *reductio ad absurdum* method to excess, a method which "convinces but does not satisfy the mind." Legendre's text is, however, left intact, alternative proofs being given in notes at the end. These alternative proofs, as well as the proofs given in the modified text of the 1876 edition, are rough applications of the theory of limits.

During the first half of the nineteenth century, and even later, the works of Legendre, Lacroix and Bézout were used extensively in France. In later editions less stress was laid upon practical applications and numerical computation. Otherwise few changes occurred. In general, school organization, based on the regulations of the time of Napoleon I., was quite fixed in France until 1870. France has a rigid centralization of authority in education. If the "Conseil d' instruction supérieure" decides upon a change, the whole country adopts it at once. As compared with the German, the French teacher has little individual freedom. France is a country with a "system of revolutions from above."* Since 1870 the movement has been toward greater individual freedom. The later tendencies in geometry are imaged in the work of Rouché and de Comberousse, which contains a large amount of new material and meets the demands of the two year courses of the *classes de mathématiques spéciales* during which as much as sixteen hours per week are given to mathematics and a degree of specialization is allowed in preparation for university courses, as in no other country. In 1902 and 1905 official courses of study were adopted in France in which greater stress is laid upon graphic representation, the idea of a variable and a function, and upon the practical applications of mathematics. This new tendency is mirrored in the geometry of E. Borel, a remarkable book, in which the practical receives due emphasis and in which intuition meets with fuller recognition. With Borel the concept of motion is prominently used. There is an introduction of eight pages on the use of the ruler, compasses, and protractor, and ten pages on the mensuration of surfaces and solids, treated empirically. Applications are skillfully interwoven with theory, throughout the book. He has well selected practical exercises involving symmetry, the nets of regular polygons, the use of pulleys, and so on. Algebraic geometry and the development of metric properties come last in the book. He introduces the rudiments of trigonometry. The usual division into plane geometry and solid geometry is not rigidly maintained.

A parallel and somewhat different tendency in France is seen in the geometry of Ch. Méray of Dijon, which was first brought out in 1874 but has only in recent years received much attention. Méray represents the severely logical mode of exposition;† he uses in his proofs no fact of observation which has not been previously set down in an axiom; he formulates a complete list of axioms, but introduces each only when it is needed;

*Klein, p. 457.
†Klein, p. 475.

nor does he aim to limit their number to a minimum. Characteristic of Méray is the complete fusion of plane and solid geometry, and the use of motion. Prepared under the influence of Méray is the recent (1908) geometry of C. Bourlet.

Influenced by the Perry movement in England and America, France is experimenting on the laboratory method of instruction.* A laboratory was founded by J. Tannery and E. Borel. Recently there has been considerable discussion in France on the question whether in laying the foundations to geometry, *motion* should be used or not. The defenders of a static theory of parallels claim that motion cannot be visualized on the board, rendering intuition more difficult. The defenders of the kinematic theory advocate the use of movable figures.†

GERMANY.

Klein‡ expresses surprise that, during the Renaissance, Euclid should have come to be looked upon as a text suitable for the first instruction in geometry. Perhaps the reason for this attitude toward Euclid lies in the fact that geometry was first taken up in the universities by students of maturer years. As geometry came gradually to be taught to younger and younger pupils, Euclid was still retained. Thus the misconception arose that Euclid was a suitable geometrical text for young boys.

While D'Alembert formulated his ideas on elementary geometry in France, A. G. Kastner evolved in Germany a type of his own, in his work, *Anfangsgründe der Arithmetik, Geometrie, Trigonometrie und Perspectiv*, Goettingen, 1758. Kastner begins with definitions and axioms in Euclidean style, develops the geometry of the plane (69 pages) and ends this part with practical applications (47 pages). The second part of the geometry begins with the geometry of space (60 pages), continues with 31 pages given to plane trigonometry and its applications to the solution of triangles, and with 9 pages of practical geometry. Then follow spherical trigonometry and 24 pages on perspective. The method of exhaustion is used. It was the opinion of Kastner that "the newer works on geometry lose the more in clearness and thoroughness, the farther they depart from Euclid." He complains that modern authors, particularly the French, have departed from the ancient rigor, "to make the study of mathematics easier for people whose main occupation is not study, namely for soldiers."

Not without interest is W. J. G. Karsten's *Lehrbegriff der gesamten Mathematik*, in eight volumes, 1767-77, the first two volumes of which are given to arithmetic and geometry. Karsten begins with arithmetic, then proceeds to plane geometry, closing with simple arithmetical applications. He proceeds thereupon to solid geometry, returns to arithmetic, and gives

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*Scholten, *Zeitschrift*, Vol. 40, 1909, pp. 444-5; *L'Enseignement Mathématique*, 11, p. 206.

†Schotten, *Zeitschrift*, Vol. 40, 1909, p. 445.

‡Klein, pp. 434, 435.

the rudiments of algebra with logarithms, followed by trigonometry and its applications to plane geometry. Finally are given the rudiments of spherical trigonometry and a fuller treatment of solids. Nowhere are heavy demands made upon the pupil. That this exposition was intended for students of university grade rather than those in the preparatory school, testifies to the low state of mathematical instruction in German universities of the eighteenth century. Close relation between arithmetic, geometry and trigonometry is also maintained in the works of J. G. Büsch (1776) and G. S. Klügel (1798), the aim being to make the subject easy of comprehension.

In the nineteenth century, until near its end, advanced mathematicians in Germany took little or no part in the improvement of the teaching of elementary mathematics. In geometry, Euclid's text was not usually taught, but the dogmatic method of Euclid was in vogue during the first half. About the middle of the century Euclid's order of the theorems came to be criticized as chaotic. It is interesting to see the Germans attack Euclid's order as arbitrary and the English defend it as the only order worthy of serious consideration. The grouping of theorems according to subjects came to be discussed in Germany.* The advocacy of object teaching by Pestalozzi, the championing of Pestalozzianism by Herbart, the attacks upon mathematical reasoning and particularly upon Euclid that were made by Schopenhauer† conspired to influence the teaching of geometry.

About 1860 the genetic method (called "heuristic" when the inventional side was emphasized) came to be discussed, which makes a plea of being a natural method, since it incites self-activity in the pupil. With the genesis of a theorem the pupil sees intuitively its inner relation to other theorems; he not only sees whence he came but also whither he is going; the reader of Euclid is blindfolded, so to speak, and then somehow transported to the next station. It is difficult to prepare text-books for the genetic method. The teacher by careful questioning one moment leads the student, the next moment follows him, and no one can foresee the exact path which this mode of advance will mark out. It is not strange, therefore, if many teachers proceeded heuristically while the texts retained mostly the dogmatic form.‡ Moreover, experience made it plain to teachers that the dogmatic statement of theorems has a high mnemotechnic value.§ While the genetic method in its pure form has not succeeded in establishing itself, it has exerted a strong influence by shifting the emphasis from the memorizing of proofs to the cultivation of originality and logical reasoning.

Another movement that sprang from the teachings of Pestalozzi and Herbart was the adoption of preliminary courses on observational geometry and drawing, about 1870. Such courses had been recommended long before this time. This movement was stronger in Germany than in England and

*Schotten, p. 11.
†Klein, p. 503.
‡Schotten, p. 96.
§Schotten, p. 13.

France. In their propædeutic courses the geometry of solids was to receive consideration and a taste of the genetic method was recommended. The pupils acquired dexterity in the use of ruler and compasses. Propædeutic courses have maintained their place to the present time.

Herbart made strong endeavors to remove the superstition that had arisen in early days when Euclid was placed in the hands of young and immature students, to the effect that mathematics could be learned only by a few pupils endowed with special gifts. According to his view the fault lies as a rule in the abstract character of the early instruction; the introduction of propædeutic courses and the greater emphasis upon "Anschauung" at all stages had shown that most students can master mathematics. Whether "amathematicians" do exist in rare instances, is a question which Klein refers to experimental psychologists for reply.*

A third movement agitated in Germany, was in favor of the introduction into elementary instruction of the concepts of the modern projective geometry. It originated about 1870.† The criticism was made that Steiner, Moebius and von Staudt had been so busy with their researches as to make no attempt to reform elementary instruction, and that text-book writers had ignored the researches of these great men. The leaders in this attempt to incorporate modern methods were Schlegel and Fiedler. A concomitant of this programme was the breaking down of the division of geometry into plane and solid, and the effort by the use of models, etc., to make geometry more concrete. To effect this reform, a number of texts by Schlegel, Müller, Kruse, Becker, Worpitzky, Henrici und Treutlein sprang into existence.‡ Aside from the production of interesting text books this agitation has had little success. The books in question were seldom used.§ Can it be that D'Alembert's dogma is, after all, based upon truth—the dogma that the historical order of development of geometry is the pedagogical order; that is, the easiest approach to the science for the young mind? Are the concepts of projective geometry more difficult to grasp than those of the older geometry, or did the texts just named overtax the pupils, and perhaps in other ways violate the demands of sound pedagogy?

Most interesting are the statistics gathered in Prussia in 1880 which showed the following distribution of geometrical texts: Kambly was used in 217 institutions; Koppe in 54; Mehler in 44; Reidt in 29, while 55 texts were used in one institution each. Kambly's "clever but unscientific book" was first issued in Breslau in 1850 and a few years ago reached the 101st edition in the revision by Roeder. Koppe was looked upon as an inferior work, yet it enjoyed great popularity. On the other hand, books like those of H. Müller and even Henrici und Treutlein seldom passed beyond the second edition. This most astonishing success of works considered as scien-

*Klein, p. 499.
†Schotten, p. 18.
‡Schotten, p. 19.
§Schotten, p. 20.

tifically inferior, requires explanation. Schlegel says* "that the quality of the books most widely adopted allows one to draw an inference respecting the scientific level of the instruction generally reached in that subject." But this remark considers merely one phase of this question. May not the mass of teachers have had a feeling or insight concerning text books which involved questions of intuition or other psychologic matters that the writers of the more scientific books overlooked? Simon† points out that until recently the German teacher, unlike the French, enjoyed complete freedom in teaching, and that small texts, like Kambly, allow his individuality much wider play.

Kambly's "Elementar-Mathematik" was made up of four parts: first, arithmetic and algebra; second, planimetry; third, plane and spherical trigonometry; fourth, stereometry. Of interest here, is the interpolation of trigonometry. We have before us Kambly's *Planimetrie,* 43d edition, Breslau, 1876. Among the points of popularity we mention the following:

1. The book contains only as much matter as a class can conveniently finish in one year. Skipping parts of a book, says Kambly, has a bad effect upon both pupils and parents.

2. The diction is clear and simple. Mathematical symbols are used freely. The setting of the type is such as to enable the eye more quickly to see the relations set forth.

3. The arrangement of the book is such as to allow the teacher much freedom. He may, for instance, omit incommensurables altogether, or else substitute for certain proofs in the regular text others given in the foot notes where rough proofs are found for incommensurable cases.

4. Easy arithmetical applications, original theorems, and original constructions are given at the end of the book, so that some, or all, may be conveniently taken or omitted, according to the preference of the teacher.

Koppe's *Planimetrie* made somewhat greater demands upon the powers of the pupil than did Kambly, but incommensurables were treated only in foot notes or in remarks following the proofs of theorems. In Lübsen's *Elementar Geometrie,* I have not been able to find a reference to incommensurables. It differs from Kambly and Koppe in having better figures and in having them on the page where they are needed, instead of the end of the book on separate sheets that unfold out. A clever feature in Lübsen are the practical applications introduced from the very beginning. How to run a straight line over undulating country by the use of poles, is explained in several diagrams on the first pages. Other figures show how to determine the distance between points on opposite banks of a river.

Since about 1890 the activity of Felix Klein of Goettingen, in mathematical reform, has been very great. For the first time since the death of Kastner, is the influence of university professors upon the teaching of ele-

*Schotten, p. 21.
†Simon, p. 25.

mentary mathematics in Germany beginning to be strongly felt. Among the defects of geometrical instruction, he points out the insufficient fusion of the various branches of elementary mathematics.* Thus, too little attention is given to drawing of solids and to projection, to the idea of motion in a figure to replace Euclidean rigidity, to the fusion of arithmetic and geometry, to the introduction of the coordinate representation of analytics. On the other hand, the construction of triangles from given data, is over emphasized,† as is also the study of the curious points and lines in the geometry of the triangle. This last criticism applies even more strongly to English text books.

Klein points out that modern demands in geometric teaching, *first*, emphasize the psychologic point of view,‡ which considers not only the subject matter, but also the pupil, and insists upon a very concrete presentation in the first stages of instruction, followed by a gradual introduction of the logical element; *second*, call for a better selection of the material from the view point of instruction as a whole; *third*, insist on a closer alignment with practical applications; *fourth*, encourage the fusion of plane and solid geometry, and of arithmetic and geometry.§

A piece of research of vital importance in the advanced study of geometry is the *Foundations of Geometry*, brought out in 1899 by Professor Hilbert of Goettingen.‖ Though widely read by mathematicians, it has exerted no direct influence upon elementary teaching in Germany. It has been felt that this mode of treatment is not suitable for pupils first entering upon demonstrative geometry.

Italy

Since the unification of Italy, great mathematical activity has existed in that country. Before that event, very different practices in geometrical teaching existed in different parts of the country.¶ In 1868 Cremona and Battaglini were members of a government commission to inquire into the state of geometrical teaching. They found it unsatisfactory, and the number of bad text books so great, and so much on the increase, that they recommended for classical schools the adoption of Euclid, an edition of which was brought out by Betti and Brioschi. Later other works of scientific merit replaced Euclid. Cremona's great emphasis upon projective geometry reached from the universities down into secondary schools. A typical work is that of A. Sannia and E. d'Ovidio, 1869, which uses the theory of limits and retains the division of geometry into plane and solid.

*Klein, p. 439.

†Klein, p. 442.

‡Klein, p. 435.

§Klein, p. 437.

‖D. Hilbert, "Grundlagen der Geometrie" in *Festschrift zur Feier der Enthüllung des Gauss-Weber Denkmals in Goettingen*, Leipzig, 1890.

¶Simon, p. 43.

It stands closer to Euclid than to Legendre. The blending of plane and solid geometry, which received great emphasis in Italy, is typified in the *Elementi di Geometria* of R. de Paolis, 1884.

A very remarkable school came into being in Italy, the purpose of which is to render geometry still more rigorous than in the Euclidean text. Starting with a single basic concept, the point, all other concepts are to be logically developed. This movement is typified in the works of G. Veronese.* Of elementary works he has prepared *Nozioni Elementari di Geometria Intuitiva*, 1902, and *Elementi di Geometria*, 1904, the first of these being a propædeutic work. Demonstrative geometry is taken up in Italy with older pupils than in Germany and the United States; hence works of greater rigor can be used. Veronese endeavors to state all the necessary postulates of geometry, no matter how obvious, as for instance, "There exist *different* points," to make it plain that we do not consider a geometry in which only one point exists.† As regards the selection of material, Veronese confines himself mainly to that of Euclid, thus receding from the tendency of the School of Cremona. He avoids all fusion with arithmetic. Somewhat similar in character is the *Elementi di Geometria* of F. Enriques and U. Amaldi, 1905.

The effort at rigor, due to Veronese, has been intensified in the great school of Peano, which endeavors to eliminate all intuition. It seems that this school has influenced even elementary instruction and the teaching in technical schools.‡ This recent Italian emphasis upon extreme rigor has led to deplorable results with the less gifted pupils, and a reaction appears to be setting in. Under the leadership of Loria and Vailati a movement is on foot favoring greater emphasis upon intuition, the introduction of some modern geometrical notions, the fusion of geometry with arithmetic, and the concession to the demands for practical applications made by this age of industrial development. In fact, Italy is entering upon a reform much like that of Germany and France.§

ENGLAND.

Roger Bacon says that toward the close of the thirteenth century the definitions and a few of the theorems in geometry were studied by some pupils at Oxford.|| About 1570 Sir Henry Savile began to lecture at Oxford on Greek geometry, and in 1619 Briggs at Cambridge on Euclid. In 1665, Isaac Barrow at Cambridge prepared a complete edition of Euclid, which was the standard for fifty years. Gow says that "The seventy years or so, from 1660 to 1730, when Wallis and Halley were professors at Oxford, Bar-

*Klein, p. 482.

†Klein, p. 483.

‡Klein, p. 486.

§For additional details see W. Lietzmann's article in Schotten's *Zeitschrift*, Vol. 39, pp. 177-191; Vol. 40, pp. 227-228.

||Ball, *Mathematics at Cambridge*, 1889, p. 3.

row and Newton at Cambridge, were the period during which the study of Greek geometry was at its height in England."* In 1703, William Whiston became the successor of Newton at Cambridge. He brought out an edition of Tacquet's Euclid. Robert Simson's edition of Euclid first appeared in 1756. Simson was professor of mathematics at the University of Glasgow. In the universities of Great Britain, Euclid met with no competition. Ward's *Young Mathematician's Guide*, 1707, may have been used to some extent, but probably more for its arithmetic and algebra than for its geometry. Practical men, holding positions as excise officers, had to be familiar with practical geometry. For them practical treatises existed, some of which gave explanations of the slide rule. A departure from Euclidean rigor might be expected in the education of men for the army or navy. We have seen that Kastner criticised the French for making mathematics easy for men interested in war. England has had since 1722 an academy at Portsmouth where men spent one or two years studying navigation, drawing, etc. England has had also, since 1741, a military academy at Woolwich, where sons of noblemen and military officers were taught fortification, gunnery and mathematics. Among the mathematical professors at Woolwich, during the eighteenth century, were Thomas Simpson, John Bonnycastle and Charles Hutton, all three authors of text books including geometries. Hutton's works went through several editions in the first half of the nineteenth century. From this it is evident that Euclid did not hold universal sway in England. Yet the forces opposing him were utterly unable to dislodge him.

In 1822, Sir David Brewster brought out an English translation of Legendre's geometry. Did teachers rally in favor of the introduction of this text? We shall see that DeMorgan suggested the use of some parts of it on solid geometry; DeMorgan deplored that solid geometry was seldom or never taught before trigonometry. But otherwise we are not able to find any serious reference to this translation of Legendre.

During the second half of the eighteenth century England had come to be the only country where Euclid was practically the only geometrical text used. During the eighteenth century the average age of freshmen in the English universities was gradually increasing, and perhaps at this time, Euclid passed from the universities to the lower schools. There is no explicit proof, however, that in the great "Public Schools" Euclid was studied before the nineteenth century.†

Very recently‡ some interesting information has been published about one of the "public schools"—Christ Hospital—which paid more than usual attention to mathematics in the courses for boys preparing to enter the royal navy. It seems that as early as 1680 such boys were required to study the earliest parts of the first book of Euclid, the 10th, 11th and 12th

*Gow, *History of Greek Geometry*, Cambridge, 1884, p. 208.

†Stamper, p. 88.

‡"A School Course in Mathematics in the XVII Century" by W. W. R. Ball in the *Mathematical Gazette*, Vol. V, 1910, Part I, pp. 202-205.

propositions of the sixth book, and to learn arithmetic. Perhaps this represented all the theoretical mathematics taught, for Sir Isaac Newton, whose advice about changes in the course was sought, notes the following omissions: There was no "symbolic arithmetic," no "taking of heights and distances and measuring of planes and solids," no "spherical trigonometry," nothing of "Mercator's chart." In other "public schools" probably no courses in geometry were given during the eighteenth century. Says Stamper: "It was not until about the middle of the nineteenth century that the study of Euclid became common in the secondary schools of England."

It would be instructive to secure more information explainiug how it was possible for Euclid to maintain its supremacy as a text, when geometry was being transferred from the universities to the schools. What were the experiences of teachers in secondary schools with the Euclidean text? The desirability of modifying Euclid must have arisen early, for in 1795 John Playfair brought out a revised Euclid containing the first six books and adding the computation of π and a book on solid geometry drawn from modern sources. Playfair endeavored to give the geometry a form which would render it more useful. Euclid's fifth book, which had never been used successfully with beginners in geometry, as far as we can ascertain, was modified by Playfair by replacing Euclid's prolix explanations by the more concise language of algebra. But Playfair did not try to change the nature of the reasoning. Had there been a strong movement against Euclid in England at this time, Playfair would probably have joined it. In his review of Leslie's *Geometry* in the Edinburgh Review, Vol. 20, 1812, p. 79, he says: "A question has been sometimes agitated whether it is most advantageous, for the study of geometry, to possess a number of elementary treatises, or to have one standard work, like that of Euclid . . . the same lessons are not suited to every intellect, and on these accounts it may be of advantage that different elementary texts should exist. We are very much inclined to the latter opinion."

William George Spencer's unique booklet on *Inventional Geometry* was brought out about 1830 or 35, but "received but little notice" at that time. A noteworthy device for aiding the young mind through sensuous stimulus was the use of colored diagrams, suggested by Oliver Byrne, in his edition of Euclid, London, 1847. The failure of this book is doubtless due to the want of moderation in the use of colors.

The ablest writer on the teaching of elementary geometry during the first half of the nineteenth century in England, was Augustus DeMorgan. His articles published in the *Quarterly Journal of Education*, in 1831, 1832, 1833, display a pedagogical insight which would have prevented many calamities in English teaching, had his views been more promptly and widely accepted. Elsewhere we quoted DeMorgan's remarks on Williamson's criticism of Clairaut's geometry, which showed that DeMorgan firmly believed

in a preliminary course in Geometry, as an introduction to a logical course like that of Euclid. It will appear that England was the last country actually to introduce propædeutic courses in elementary instruction.

DeMorgan did not hesitate to recommend radical changes in Euclid. Here is what he said in 1831, in an article in the *Quarterly Journal of Education*, entitled "On Mathematical Instruction:"

"With regard to the fifth book of the Elements, we recommend the teacher to substitute for it the common arithmetical notions of proportion. Admitting that this is not so exact as the method of Euclid, still, a less rigorous but intelligible process is better than a perfect method which cannot be understood by the great majority of learners. The sixth book would thus become perfectly intelligible."

Two years later, in an article in the same journal "On the Methods of Teaching the Elements of Geometry," DeMorgan dares to suggest that certain parts of Legendre might be profitably substituted for parts of Euclid. "The eleventh book of Euclid may, in our opinion, be abandoned with advantage in favour of more modern works on solid geometry, particularly that of Legendre, which the English reader will find in Sir David Brewster's Translation." In the same article DeMorgan gives utterance to a difficulty experienced by young students, which has been referred to by many writers in different countries, the *reductio ad absurdum.* DeMorgan says: "The most serious embarrassment in the purely reasoning part is the *reductio ad absurdum,* or indirect demonstration. This form of argument is generally the last to be clearly understood, though it occurs almost on the threshold of the elements. We may find the key to the difficulty in the confined ideas which prevail on the modes of speech there employed." As regards the difficult fifth book, DeMorgan said, in 1833, "We would say to all, teach the fifth book, *if you can;* but we would have all remember that there is an *if.*" In another place he adds: "We strongly suspect that Euclid, as studied, does as much harm as good." To the credit of teachers be it said, that the fifth book was quite generally omitted. But DeMorgan's activity in this line did not end here. In 1836 he published *The Connexion of Number and Magnitude; An attempt to explain the fifth book of Euclid.* For fifty years this tract was not duly appreciated; later it began to wield a wide influence; it is on this tract that the substitute for the fifth book given in the Syllabus of the Association for the Improvement of Geometrical Teaching is modeled; it is on this tract that the revised fifth book in the more recent editions of Euclid by Nixon and by Hall and Stevens is based.

The need of modifying the text of Euclid is brought out by DeMorgan in the *Companion to the British Almanac* of 1849, page 20, as follows: "If the study of Euclid have been almost abandoned on the Continent, and have declined in England, it is because his more ardent admirers have insisted on regarding the accidents of his position as laws of the science."

How little influence DeMorgan's views wielded in England before

about 1870 as regards the revision of Euclid's fifth book and the study of solid geometry, appears from the fact that the most popular edition of Euclid for many years, was the one brought out in 1862 by Todhunter. This author reproduces Simson's text, though he greatly assists the pupil in overcoming the difficulties by breaking up the demonstrations into their constituent parts. In an Appendix are given notes, supplementary propositions and original exercises. Todhunter was quite out of sympathy with the purposes of the Association for the Improvement of Geometrical Teaching.*

Opponents of Euclid existed in England at all times. Thus in 1860 W. D. Cooley brought out a rival text. Eleven years later he expressed himself regarding this venture as follows:†

"In 1860 there was published for me, by Messrs. Williams and Norgate, a little volume entitled, *The Elements of Geometry Simplified and Explained,* adapted to the system of empirical proof, and of exhibiting the truth of theorems by means of figures cut in paper. It contains in 35 theorems the quintessence of Euclid's first six books, together with a supplement not in Euclid. There was no gap in the sequence or chain of reasoning, yet the 32d and 47th propositions of Euclid were, respectively, the 3d and 17th of my series. This book proved a failure, for which several reasons might be given, but it will be sufficient here to state but one, namely, that it came forth ten years before its time."

The reformers found a champion in Sylvester, who in 1869, before Section A of the British Association exclaimed: "I should rejoice to see mathematics taught with that life and animation which the presence and example of her young and buoyant sister (natural science) could not fail to impart, short roads preferred to long ones, Euclid honourably shelved, or buried 'deeper than e'er plummet sounded' out of the school boy's reach. . . ." The reform forces finally organized themselves, in 1871, into the "Association for the Improvement of Geometrical Teaching" (A. I. G. T.).

It is a curious circumstance that England's great mathematician, Arthur Cayley, opposed this reform movement. His admiration for Euclid was so ardent that he even expressed a preference for the original treatise without Simson's additions. In the opinion of Langley, Cayley "overshot the mark and his opposition told in favor of the Association."‡

The second *Report* of the A. I. G. T. recommended practical exercises in geometrical construction, easy originals and numerical examples. Two years were given to the preparation of the Geometrical Syllabus on proportion. A double syllabus was prepared: A Syllabus on Geometric Constructions, and a Syllabus on Plane Geometry. Most of DeMorgan's suggestions§ on the revision of the fifth book of Euclid were adopted.

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*See *Conflict of Studies,* by Todhunter, London, 1873.
†Nature, Vol. 4, 1871, p. 486.
‡*Fifth Report of the A. I. G. T.,* p. 21.
§*Companion to the British Almanac,* 1849, pp. 5-20.

This Society, after long labors, finally issued a substitute text, *The Elements of Plane Geometry.* This was not used at home, but was used with success in the British colonies. Klein expresses himself, as follows, in regard to it: "This is essentially merely a smoothed down and polished presentation of the first six books of Euclid's elements; thus the rough places at the beginning of the first book . . are removed by a consistent use of the concept of motion, but in general the sequence and the contents of Euclid are adhered to, in deference to the examinations. It is therefore only a tame reform, that is here attempted; nevertheless, it has met with sharp opposition by the adherents of the old English system. As proof of this, I refer to an amusingly written book of Dodgson, *Euclid and His Modern Rivals.*" Here Euclid comes out victorious, and all reformers, particularly Legendre and members of the A. I. G. T., are put to the rout. J. M. Wilson's *Elementary Geometry*, 1st edition, 1869, came in for a large share of the criticism. At Oxford, where Dodgson had given instruction in geometry for many years, this same Wilson had, at one time, read a critical paper before the Mathematical Society, on "Euclid as a Text Book of Elementary Geometry." Wilson was a prime mover in the organization of the A. I. G. T.

Since about 1870, many editions of Euclid have been printed containing revisions with the object of better adapting Euclid to school use. They exhibit all possible gradations of departure from the original text. There appeared sequels to Euclid like that of F. Casey. Professor Klein expresses himself in regard to these as follows: "The necessity has been felt to consider modern research, going beyond Euclid; this has been done by pressing it by force into the rigid Euclidean form, whereby a good part of the modern spirit is, of course, lost."*

During thirty years the A. I. G. T. appeared to have accomplished comparatively little. It had secured the concession that proofs different from Euclid's shall be accepted in examinations and had brought about a sentiment favoring some modification and enrichment of the Euclidean text. In reality, it had accomplished much more, for it had prepared the way for the great agitation of 1901, known as the Perry Movement, which called for a complete divorce from Euclid. The discussion of the teaching of mathematics at the Glasgow meeting of the British Association marks an epoch. The following are suggestions and criticisms that were contained in Perry's Syllabus.†

1. Experimental geometry and practical mensuration to precede demonstrative geometry. Use of squared paper. Rough guessing at lengths and weights to be encouraged.
2. Some deductive reasoning to accompany experimental geometry.
3. More emphasis on solid geometry; this subject has been postponed too long.

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*Klein, p. 447.

†*Discussion on the Teaching of Mathematics*, edited by John Perry, 1901, p. 97.

4. Adoption of coordinate representation in space.
5. The introduction of trigonometric functions in the study of geometry.
6. Emphasis upon the utilitarian parts of the subject.
7. Examinations conducted by any other examiner than the pupil's teacher are imperfect examinations.

In criticism of previous practices, Perry held that a boy should be educated through the experiences he already possesses, and should be allowed to assume the truth of many propositions. He held that the teacher must recognize that boys take unkindly to abstract reasoning. He criticised Oxford because, for the pass degree there, two books of Euclid must be memorized, even including the lettering of figures, no original exercises being required. In the discussion that followed, all favored the preliminary experimental course and some advocated a second experimental course to accompany Euclid. Hudson and Forsyth still believed in maintaining the Euclidean sequence of theorems. Minchin declared Euclid's order bad. S. P. Thompson and MacMahon favored the retention of Euclid. Miall did not see why we should have a recognized geometry any more than one arithmetic, or one trigonometry. Minchin, Magnus, Pressland, Workman and Lamb declared themselves against Euclid as a text book.

The immediate result of Perry's address of 1900, at Glasgow, was the appointment of two committees, one of the British Association and the other of the Mathematical Association. The former committee confined its work to the more general aspects of geometrical teaching. The latter, which was composed mainly of school masters, formulated a set of detailed recommendations, which were published in the *Mathematical Gazette* of May, 1902. They include an experimental introductory course, requiring the use of instruments, practical measurement and numerical work. In the formal study of geometry is recommended the retention of Euclid as a framework, the admission of hypothetical constructions, definitions not to be taught *en bloc*, the omission of incommensurables in the ordinary school course, the use of algebra in the treatment of areas.

The Perry laboratory method has led to the preparation of some severely practical works, but as Lodge says, Perry "over emphasized fact divorced from principles." A middle ground has met with greater favor. The plans recommended by the Mathematical Association have been embodied very successfully by Godfrey and Siddons in a text book entitled, *Elementary Geometry, Practical and Theoretical* (Cambridge University Press, 1904). The recommendations of the Mathematical Association have met with favor among teachers, and the general effect has been beneficial. A circular issued in 1908-1909, by the Board of Education, on *The Teaching of Geometry and Graphic Algebra*, showed the wide departure made since the beginning of the twentieth century. We quote two sentences: "Axioms and postulates should not be learnt or even mentioned." "It

should be frankly recognized that unless the power of doing riders has been developed, the study of the subject is a failure."*

The greatest obstacle to reform in England has been the system of examinations. After thirty years of failures the Mathematical Association, at last, has been remarkably successful in persuading examining bodies to give up their insistence upon Euclid, and now Euclid's proofs and arrangement are no longer required by the universities. "Any proof of a proposition will be accepted which appears to the examiners to form a part of a logical order of treatment."

THE UNITED STATES OF AMERICA.

During the seventeenth century, arithmetic and geometry received some attention in the last year of the college course at Harvard College. In 1726 Alsted's *Geometry* is mentioned as a text book studied by Harvard seniors, but as soon as geometry came to receive serious attention in American colleges, Euclid became the text used. The first mention of Euclid that we have seen, at Yale, is in 1733; at Harvard, in 1737. In the latter part of the eighteenth century, geometry was taught to lower classmen. According to a member of the Harvard class of 1798, "the sophomore year gave us Euclid to measure our strength." In 1801 Professor Webber said, "A tutor teaches in Harvard College Playfair's Elements of Geometry."

In 1813 the "Analytical Society" was formed at Cambridge in England, which aimed to encourage in Britain the vigorous study of French higher mathematics. The influence of this movement reached the United States. In about ten years American teachers began to adopt French texts. Collateral events at West Point had the same tendency. There elementary mathematics was taught from 1808 to 1810 by F. R. Hassler, who was a graduate of the University of Berne in Switzerland. In 1817 Crozet, of the Polytechnic School in Paris, introduced descriptive geometry into West Point.

In 1819, John Farrar, of Harvard, brought out a translation of Legendre's Geometry, which, with translations made by him of other French and Swiss texts on mathematics, were at once widely adopted in the leading American colleges. American teachers were willing to turn to the French, not only for works on the calculus and celestial mechanics, but also for books on elementary mathematics. So it came about that Euclid was replaced by Legendre. In 1828 Charles Davies, professor at West Point, brought out an edition of Brewster's translation of Legendre's Geometry. Davies did not enunciate propositions with reference to and by the aid of the particular diagram used for the demonstration, and to that extent returned to the method of Euclid. Davies' edition became widely popular under the name of "Davies-Legendre," and was much used in the United States as late as the 70's.

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*Nature, Vol. 80, 1909, May 27, p. 374.

One of the earliest American geometries worthy of note, was that of Benjamin Peirce. The Harvard catalogue of 1838 announces that Freshmen take Peirce's Geometry. Peirce favored the use of infinitesimals and also the use of the term direction, a concept probably first used in this country by a Harvard teacher named Hayward in his geometry of 1829. Peirce's text did not become widely popular, for, like his other elementary books, it was too condensed for immature students. In 1843 or 1844, Harvard first made geometry a requirement for admission to College.

In 1851, Professor Elias Loomis, of Yale, issued a geometry which was revised in 1871. Loomis came under French influences as a student in Paris. In the second edition of his text he says: "The present volume follows substantially the order of Blanchet's Legendre, while the form of the demonstration is modeled after the more logical method of Euclid." It has been said of American writers, that while they have given up Euclid, they have modified Legendre's Geometry so as to make it resemble Euclid as much as possible. This applies to Loomis with greater force perhaps than to any other author.

In 1871 Professor Olney, of the University of Michigan, published a Geometry under two main heads:

I. *Special or Elementary Geometry,* comprising (1) Empirical Geometry, (2) Demonstrative Geometry, (3) Original Exercises in the Application of Algebra to Geometry, (4) Trigonometry.

II. *General Geometry* (Plane Loci).

Olney was a self educated man. He was a great teacher and had original ideas about teaching. It is said that he was prevented by his publishers from departing very far from the traditional classification. His ideas were novel and forecasted in many ways the present tendencies in mathematical teaching. His geometry shows that he attempted to correlate the various mathematical topics and to introduce applications to every day affairs. Olney's books were used quite extensively in the Middle West, but acquired no firm foothold in the East.

Just before the death of William Chauvenet, in 1870, appeared his *Geometry,* the only elementary book he wrote. Closely following French models, exhibiting a wonderful ease and grace of style, Chauvenet produced a remarkable book, which was used in many of the best schools. He included as a part of the work, an introduction to modern geometry. Perhaps no work on geometry ever published in the United States has been so highly respected as this.

In 1878 appeared the geometry of G. A. Wentworth, which is still in use. We omit all discussion of it, as also of later books which have been published in this country.

The researches on non-Euclidean Geometry, begun in the eighteenth century in Italy and Germany, and brought to fruition in the early part of the nineteenth century, did not produce appreciable effect upon the teaching

of elementary geometry until the last quarter of the nineteenth century. It was in 1867 and 1868 that Baltzer, Battaglini, Grunert and Hoüel brought Bolyai and Lobatchevsky to the attention of the mathematical public at large.

The new ideas have not affected the teaching of elementary geometry except in some of the definitions and postulates. They have assisted in the rejection of the definitions, "parallel lines are lines everywhere equally distant," and "parallel lines are straight lines which have the same direction." They have shown the futility of "proving" the parallel-postulate and have led to the use of the word "axiom," not as a "self-evident truth," but as a synonym for "postulate."

In conclusion, we note that, with the beginning of the twentieth century, England began once more to influence the teaching of geometry in the United States, through the so-called "Perry movement," and that Germany, which at no time during the nineteenth century affected geometrical teaching in America, makes itself felt at the present time through the pupils of Klein and Hilbert and through the international movement towards reform in the teaching of mathematics, headed by Klein.

ON THE SOLUTIONS OF A SYSTEM OF LINEAR EQUATIONS.

By G. A. MILLER, University of Illinois.

In the June-July number of this journal we proved the theorem that the necessary and sufficient condition that a given unknown has the same value in every solution of a consistent system of linear equations is that the rank of the matrix is reduced by omitting the coefficients of this unknown from this matrix. The following proof of this important theorem is probably more satisfactory to many readers. Suppose that the omission of the coefficients of x_1 reduces the rank of the matrix of the following consistent system of m equations in n unknowns:

$$
\text{(A)} \qquad \begin{array}{l}
a_{11}\, x_1 + a_{12}\, x_2 + \ldots + a_{1n}\, x_n + k_1 = 0, \\
a_{21}\, x_1 + a_{22}\, x_2 + \ldots + a_{2n}\, x_n + k_2 = 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{m1}\, x_1 + a_{m2}\, x_2 + \ldots + a_{mn}\, x_n + k_n = 0.
\end{array}
$$

That is, if r is the rank of the matrix of this system, and if we let